

INEQUALITIES FOR THE MODIFIED k - BESSEL FUNCTION

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ABSTRACT. The article considers the generalized k -Bessel functions and represents it as Wright functions. Then we study the monotonicity properties of the ratio of two different orders k - Bessel functions, and the ratio of the k -Bessel and the m -Bessel functions. The log-convexity with respect to the order of the k -Bessel also given. An investigation regarding the monotonicity of the ratio of the k -Bessel and k -confluent hypergeometric functions are discussed.

1. INTRODUCTION

One of the generalization of the classical gamma function Γ studied in [4] is defined by the limit formula

$$\Gamma_k(x) := \lim_{n \rightarrow \infty} \frac{n! k^n (n^k)^{\frac{x}{k}-1}}{(x)_{n,k}}, \quad k > 0, \quad (1.1)$$

where $(x)_{n,k} := x(x+k)(x+2k)\dots(x+(n-1)k)$ is called k -Pochhammer symbol. The above k -gamma function also have an integral representation as

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt, \quad \operatorname{Re}(x) > 0. \quad (1.2)$$

Properties of the k -gamma functions have been studies by many researchers [6, 8–11]. Follwoing properties are required in sequel:

- (i) $\Gamma_k(x+k) = x\Gamma_k(x)$
- (ii) $\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right)$
- (iii) $\Gamma_k(k) = 1$
- (iv) $\Gamma_k(x+nk) = \Gamma_k(x)(x)_{n,k}$

Motivated with the above generalization of the k -gamma functions, Romero et. al.[1] introduced the k -Bessel function of the first kind defined by the series

$$J_{k,\nu}^{\gamma,\lambda}(x) := \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n + \nu + 1)} \frac{(-1)^n (x/2)^n}{(n!)^2}, \quad (1.3)$$

where $k \in \mathbb{R}^+$; $\alpha, \lambda, \gamma, \nu \in C$; $\operatorname{Re}(\lambda) > 0$ and $\operatorname{Re}(\nu) > 0$. They also established two recurrence relations for $J_{k,\nu}^{\gamma,\lambda}$.

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In this article, we are considering the following function:

$$I_{k,\nu}^{\gamma,\lambda}(x) := \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n + \nu + 1)} \frac{(x/2)^n}{(n!)^2}, \quad (1.4)$$

Since

$$\lim_{k,\lambda,\gamma \rightarrow 1} I_{k,\nu}^{\gamma,\lambda}(x) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n + \nu + 1)} \frac{(x/2)^n}{n!} = \left(\frac{2}{x}\right)^{\frac{\nu}{2}} I_{\nu}(\sqrt{2x}),$$

the classical modified Bessel functions of first kind. In this sense, we can call $I_{k,\nu}^{\gamma,\lambda}$ as the modified k -Bessel functions of first kind. In fact, we can express both $J_{k,\nu}^{\gamma,\lambda}$ and $I_{k,\nu}^{\gamma,\lambda}$ together in

$$W_{k,\nu,c}^{\gamma,\lambda}(x) := \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n + \nu + 1)} \frac{(-c)^n (x/2)^n}{(n!)^2}, \quad c \in \mathbb{R}. \quad (1.5)$$

We can termed $W_{k,\nu}^{\gamma,\lambda}$ as the generalized k -Bessel function.

First we study the representation formulas for $W_{k,\nu}^{\gamma,\lambda}$ in term of the classical Wright functions. Then we will study about the monotonicity and log-convexity properties of $I_{k,\nu}^{\gamma,\lambda}$.

2. REPRESENTATION FORMULA FOR THE GENERALIZED k -BESSEL FUNCTION

The generalized hypergeometric function ${}_pF_q(a_1, \dots, a_p; c_1, \dots, c_q; x)$, is given by the power series

$${}_pF_q(a_1, \dots, a_p; c_1, \dots, c_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(c_1)_k \cdots (c_q)_k (1)_k} z^k, \quad |z| < 1, \quad (2.1)$$

where the c_i can not be zero or a negative integer. Here p or q or both are allowed to be zero. The series (2.1) is absolutely convergent for all finite z if $p \leq q$ and for $|z| < 1$ if $p = q + 1$. When $p > q + 1$, then the series diverge for $z \neq 0$ and the series does not terminate.

The generalized Wright hypergeometric function ${}_p\psi_q(z)$ is given by the series

$${}_p\psi_q(z) = {}_p\psi_q \left[\begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^p \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^q \Gamma(b_j + \beta_j k)} \frac{z^k}{k!}, \quad (2.2)$$

where $a_i, b_j \in \mathbb{C}$, and real $\alpha_i, \beta_j \in \mathbb{R}$ ($i = 1, 2, \dots, p; j = 1, 2, \dots, q$). The asymptotic behavior of this function for large values of argument of $z \in \mathbb{C}$ were studied in [13, 14] and under the condition

$$\sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i > -1 \quad (2.3)$$

in literature [18, 19]. The more properties of the Wright function are investigated in [14–16].

Now we will give the representation of the generalized k -Bessel functions in terms of the Wright and generalized hypergeometric functions.

Proposition 2.1. *Let, $k \in \mathbb{R}$ and $\lambda, \gamma, \nu \in \mathbb{C}$ such that $\operatorname{Re}(\lambda) > 0, \operatorname{Re}(\nu) > 0$. Then*

$$\mathbb{W}_{k,\nu,c}^{\gamma,\lambda}(x) = \frac{1}{k^{\frac{\nu+k+1}{k}} \Gamma\left(\frac{\gamma}{k}\right)} {}_1\psi_2 \left[\begin{matrix} \left(\frac{\gamma}{k}, 1\right) \\ \left(\frac{\nu+1}{k}, \frac{\gamma}{k}\right) \end{matrix} \middle| -\frac{cx}{2k^{\frac{\lambda}{k}-1}} \right]$$

Proof. Using the relations $\Gamma_k(x) = k^{\frac{x}{k}-1} \Gamma\left(\frac{x}{k}\right)$ and $\Gamma_k(x+nk) = \Gamma_k(x)(x)_{n,k}$, the generalized k -Bessel functions defined in (1.5) can be rewrite as

$$\mathbb{W}_{k,\nu,c}^{\gamma,\lambda}(x) = \sum_{n=0}^{\infty} \frac{\Gamma_k(\gamma+nk)}{\Gamma_k(\lambda n + \nu + 1) \Gamma_k(\gamma)} \frac{(-c)^n}{(n!)^2} \left(\frac{x}{2}\right)^n \quad (2.4)$$

$$= \frac{1}{k^{\frac{\nu+k+1}{k}} \Gamma\left(\frac{\gamma}{k}\right)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\gamma}{k} + n\right)}{\Gamma\left(\frac{\lambda}{k}n + \frac{\nu+1}{k}\right) \Gamma\left(\frac{\gamma}{k}\right)} \frac{(-c)^n}{\Gamma(n+1)\Gamma(n+1)} \left(\frac{x}{2k^{\frac{\lambda}{k}-1}}\right)^n \quad (2.5)$$

$$= \frac{1}{k^{\frac{\nu+k+1}{k}} \Gamma\left(\frac{\gamma}{k}\right)} {}_1\psi_2 \left[\begin{matrix} \left(\frac{\gamma}{k}, 1\right) \\ \left(\frac{\nu+1}{k}, \frac{\gamma}{k}\right) \end{matrix} \middle| -\frac{cx}{2k^{\frac{\lambda}{k}-1}} \right] \quad (2.6)$$

Hence the result follows. \square

3. MONOTONICTY AND LOG-CONVEXITY PROPERTIES

This section discuss the monotonicity and log-convexity properties for the modified k -Bessel functions $\mathbb{W}_{k,\nu,-1}^{\gamma,\lambda}(x) = \mathbb{I}_{k,\nu}^{\gamma,\lambda}(x)$.

Following lemma due to Biernacki and Krzyż [7] will be required.

Lemma 3.1. [7] *Consider the power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$, where $a_k \in \mathbb{R}$ and $b_k > 0$ for all k . Further suppose that both series converge on $|x| < r$. If the sequence $\{a_k/b_k\}_{k \geq 0}$ is increasing (or decreasing), then the function $x \mapsto f(x)/g(x)$ is also increasing (or decreasing) on $(0, r)$.*

The above lemma still holds when both f and g are even, or both are odd functions.

Theorem 3.1. *The following results holds true for the modified k -Bessel functions.*

- (1) *For $\mu \geq \nu > -1$, the function $x \mapsto \mathbb{I}_{k,\mu}^{\gamma,\lambda}(x)/\mathbb{I}_{k,\nu}^{\gamma,\lambda}(x)$ is increasing on $(0, \infty)$ for some fixed $k > 0$.*
- (2) *If $k \geq \lambda \geq m > 0$, the function $x \mapsto \mathbb{I}_{k,\nu}^{\gamma,\lambda}(x)/\mathbb{I}_{m,\nu}^{\gamma,\lambda}(x)$ is increasing on $(0, \infty)$ for some fixed $\nu > -1$ and $\gamma \geq \nu + 1$.*
- (3) *The function $\nu \mapsto \mathbb{I}_{k,\nu}^{\gamma,\lambda}(x)$ is log-convex on $(0, \infty)$ for some fixed $k, \gamma > 0$ and $x > 0$. Here, $\mathbb{I}_{k,\nu}^{\gamma,\lambda}(x) := \Gamma_k(\nu + 1) \mathbb{I}_{k,\nu}^{\gamma,\lambda}(x)$.*
- (4) *Suppose that $\lambda \geq k > 0$ and $\nu > -1$. Then*
 - (a) *The function $x \mapsto \mathbb{I}_{k,\nu}^{\gamma,\lambda}(x)/\Phi_k(a, c; x)$ is decreasing on $(0, \infty)$ for $a \geq c > 0$ and $0 < \gamma \leq \nu + 1$. Here, $\Phi_k(a, c; x)$ is the k -confluent hypergeometric functions.*
 - (b) *The function $x \mapsto \mathbb{I}_{k,\nu}^{\gamma,\lambda}(x)/\Phi_k(\gamma; \lambda; x/2)$ is decreasing on $(0, 1)$ for $\gamma > 0$ and $0 < k \leq \lambda \leq \nu + 1$.*
 - (c) *The function $x \mapsto \mathbb{I}_{k,\nu}^{\gamma,\lambda}(x)/\Phi_k(\gamma; \lambda; x/2)$ is decreasing on $[1, \infty)$ for $\gamma > 0$ and $0 < k \leq \min\{\lambda, \nu + 1\}$.*

Proof. (1) Form (1.4) it follows that

$$\mathbb{I}_{k,\nu}^{\gamma,\lambda}(x) = \sum_{n=0}^{\infty} a_n(\nu)x^n \quad \text{and} \quad \mathbb{I}_{k,\nu}^{\gamma,\lambda}(x) = \sum_{n=0}^{\infty} a_n(\mu)x^n,$$

where

$$a_n(\nu) = \frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n + \nu + 1)(n!)^2 2^n} \quad \text{and} \quad a_n(\mu) = \frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n + \mu + 1)(n!)^2 2^n}$$

Consider the function

$$f(t) := \frac{\Gamma_k(\lambda t + \mu + 1)}{\Gamma_k(\lambda t + \nu + 1)}.$$

Then the logarithmic differentiation yields

$$\frac{f'(t)}{f(t)} = \lambda(\Psi_k(\lambda t + \mu + 1) - \Psi_k(\lambda t + \nu + 1)).$$

Here, $\Psi_k = \Gamma'_k/\Gamma_k$ is the k -digamma functions studied in [5] and defined by

$$\Psi_k(t) = \frac{\log(k) - \gamma_1}{k} - \frac{1}{t} + \sum_{n=1}^{\infty} \frac{t}{nk(nk + t)} \quad (3.1)$$

where γ_1 is the Euler-Mascheronis constant.

A calculation yields

$$\Psi'_k(t) = \sum_{n=0}^{\infty} \frac{1}{(nk + t)^2}, \quad k > 0 \quad \text{and} \quad t > 0. \quad (3.2)$$

Clearly, Ψ_k is increasing on $(0, \infty)$ and hence $f'(t) > 0$ for all $t \geq 0$ if $\mu \geq \nu > -1$. This, in particular, implies that the sequence $\{d_n\}_{n \geq 0} = \{a_n(\nu)/a_n(\mu)\}_{n \geq 0}$ is increasing and hence the conclusion follows from Lemma 3.1.

(2). This result also follows from Lemma 3.1 if the sequence $\{d_n\}_{n \geq 0} = \{a_n^k(\nu)/a_n^m(\mu)\}_{n \geq 0}$ is increasing for $k \geq m > 0$. Here,

$$a_n^k(\nu) = \frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n + \nu + 1)(n!)^2} \quad \text{and} \quad a_n^m(\nu) = \frac{(\gamma)_{n,m}}{\Gamma_m(\lambda n + \nu + 1)(n!)^2},$$

which together with the identity $\Gamma_k(x + nk) = \Gamma_k(x)(x)_{n,k}$ gives

$$\begin{aligned} d_n &= \frac{(\gamma)_{n,k}}{(\gamma)_{n,m}} \frac{\Gamma_m(\lambda n + \nu + 1)}{\Gamma_k(\lambda n + \nu + 1)} \\ &= \frac{\Gamma_k(\gamma + nk) \Gamma_m(\lambda n + \nu + 1)}{\Gamma_k(\gamma + nm) \Gamma_k(\lambda n + \nu + 1)}. \end{aligned}$$

Now to show that $\{d_n\}$ is increase, consider the function

$$f(y) := \frac{\Gamma_k(\gamma + yk) \Gamma_m(\lambda y + \nu + 1)}{\Gamma_k(\gamma + ym) \Gamma_k(\lambda y + \nu + 1)}$$

The logarithmic differentiation of f yields

$$\frac{f'(y)}{f(y)} = k\Psi_k(\gamma + yk) + \lambda\Psi_m(\lambda y + \nu + 1) - m\Psi_m(\gamma + ym) - \lambda\Psi_k(\lambda y + \nu + 1) \quad (3.3)$$

If $\gamma \geq \nu + 1$ and $k \geq \lambda \geq m$, then (3.3) can be rewrite as

$$\frac{f'(y)}{f(y)} \geq \lambda(\Psi_k(\nu + 1 + yk) - \Psi_k(\lambda y + \nu + 1)) + m(\Psi_m(\lambda y + \nu + 1) - \Psi_m(\nu + 1 + ym)) \geq 0. \quad (3.4)$$

This conclude that f , and consequently the sequence $\{d_n\}_{n \geq 0}$, is increasing. Finally the result follows from the Lemma 3.1.

(3). It is known that sum of the log-convex functions is log-convex. Thus, to prove the result it is enough to show that

$$\nu \mapsto a_n^k(\nu) := \frac{(\gamma)_{n,k} \Gamma_k(\nu + 1)}{\Gamma_k(\lambda n + \nu + 1) (n!)^2}$$

is log-convex.

A logarithmic differentiation of $a_n(\nu)$ with respect to ν yields

$$\frac{\partial}{\partial \nu} \log(a_n^k(\nu)) = \Psi_k(\nu + 1) - \Psi_k(\lambda n + \nu + 1).$$

This along with (3.2) gives

$$\begin{aligned} \frac{\partial^2}{\partial \nu^2} \log(a_n^k(\nu)) &= \Psi'_k(\nu + 1) - \Psi'_k(\lambda n + \nu + 1) \\ &= \sum_{r=0}^{\infty} \frac{1}{(rk + \nu + 1)^2} - \sum_{r=0}^{\infty} \frac{1}{(rk + \lambda n + \nu + 1)^2} \\ &= \sum_{r=0}^{\infty} \frac{\lambda n(2rk + \lambda n + 2\nu + 2)}{(rk + \nu + 1)^2 (rk + \lambda n + \nu + 1)^2} > 0, \end{aligned}$$

for all $n \geq 0$, $k > 0$ and $\nu > -1$. Thus, $\nu \mapsto a_n^k(\nu)$ is log-convex and hence the conclusion.

(4). Denote $\Phi_k(a, c; x) = \sum_{n=0}^{\infty} c_{n,k}(a, c)x^n$ and $\mathcal{I}_{k,\nu}^{\gamma,\lambda}(x) = \sum_{n=0}^{\infty} a_n(\nu)x^n$, where

$$a_n(\nu) = \frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n + \nu + 1)(n!)^2 2^n} \quad \text{and} \quad d_{n,k}(a, c) = \frac{(a)_{n,k}}{(c)_{n,k} n!}$$

with $v > -1$ and $a, c, \lambda, \gamma, k > 0$. To apply Lemma 3.1, consider the sequence $\{w_n\}_{n \geq 0}$ defined by

$$\begin{aligned} w_n = \frac{a_n(\nu)}{d_{n,k}(a, c)} &= \frac{\Gamma_k(\gamma + nk)}{2^n \Gamma_k(\gamma) \Gamma_k(\lambda n + \alpha + 1) (n!)^2} \cdot \frac{\Gamma_k(a) \Gamma_k(c + nk) n!}{\Gamma_k(a + nk) \Gamma_k(c)} \\ &= \frac{\Gamma_k(a)}{\Gamma_k(\gamma) \Gamma_k(c)} \rho_k(n) \end{aligned}$$

where

$$\rho_k(x) = \frac{\Gamma_k(\gamma + xk) \Gamma_k(c + xk)}{\Gamma_k(\lambda x + \nu + 1) \Gamma_k(a + xk) 2^x \Gamma(x + 1)}.$$

In view of the increasing properties of Ψ_k on $(0, \infty)$, and

$$\frac{\rho'(x)}{\rho(x)} = k\psi_k(\gamma + xk) + k\psi_k(c + xk) - \lambda\psi_k(\lambda x + \alpha + 1) - k\psi_k(a + xk),$$

it follows that for $a \geq c > 0$, $\lambda \geq k$ and $\nu + 1 \geq \gamma$, the function ρ is decreasing on $(0, \infty)$ and thus the sequence $\{w_n\}_{n \geq 0}$ also decreasing. Finally the conclusion for (a) follows from the Lemma 3.1.

In the case (b) and (c), the sequence $\{w_n\}$ reduces to

$$w_n = \frac{a_n(\nu)}{d_{n,k}(\gamma, \lambda)} = \frac{\rho_k(n)}{\Gamma_k(\lambda)}$$

where

$$\rho_k(x) = \frac{\Gamma_k(\lambda + xk)}{\Gamma_k(\nu + 1 + \lambda x)\Gamma(x + 1)}.$$

Now as in the proof of part (a)

$$\frac{\rho'_k(x)}{\rho_k(x)} = k\Psi_k(\lambda + xk) - \lambda\Psi_k(\nu + 1 + xk) - \Psi(x + 1) > 0,$$

if $\nu + 1 + \lambda x \geq \lambda + xk$. Now for $x \in (0, 1)$, this inequality holds if $0 < k \leq \lambda \leq \nu + 1$, while for $x \geq 1$, it is required that $k \leq \min\{\lambda, \nu + 1\}$. \square

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